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J. Phys. A: Math. Gen. **20** (1987) 3185-3197. Printed in the UK

An ergodic theorem for intermittency of piecewise linear iterated maps

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Received 17 March 1986

Abstract. Intermittency behaviour occurs and can be exactly calculated for iterated asymmetric tent maps. A typical trajectory exhibits long regular phases with monotonic growth (according to a power law) which are interrupted by short irregular bursts at apparently random times. The lengths of the laminar phases in a long trajectory turn out to be geometrically distributed and to be independent. They are governed by their own ergodic theorem. The results can be applied to a larger class of one-hump maps.

1. Introduction

Intermittency is a well known phenomenon in data obtained experimentally and in numerical simulations of dynamical systems. Long phases of regular behaviour are interrupted by short irregular bursts at seemingly random times. The question arises of how to describe the randomness of the interruptions from a probabilistic point of view although they originate in pure deterministic dynamics. In other words, if the sequence of the lengths of the laminar phases is given, how can we characterise this apparently random sequence of natural numbers?

For iterated maps of an interval into itself, $x_{n+1}=f(x_n)$, the occurrence of intermittency can be understood by the fact that the function $f(x)$ comes very near to the straight line $y=x$ in some subinterval. Therefore the motion in this 'tube' is laminar for a long time, whereas outside larger jumps and reinjection occur. Pomeau and Manneville (1980) and Hirsch *et al* (1982) analysed this phenomenon in connection with tangent bifurcations and estimated the mean duration of the laminar phase. However, assumptions about the distribution of the initial values of the laminar phases must be made, because the latter are not exactly known in general. Moreover, correlations of the lengths of different phases cannot be calculated.

In the following we want to introduce a simple model which allows an exactly calculable solution.

2. The model

Let us consider iterated maps

$$x_{n+1}=f(x_n) \qquad n=0,1,2,\ldots \tag{1}$$

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of the interval [0, 1] onto itself, where f is taken to be a continuous piecewise linear function with one maximum (figure 1):

$$f(x)=\begin{cases}\alpha x & 0\leqslant x\leqslant 1/\alpha,\ \alpha>1\\ [\alpha/(\alpha-1)](1-x) & 1/\alpha<x\leqslant 1,\ \alpha>1.\end{cases} \tag{2}$$

For the sake of brevity we use

$$\begin{aligned} f_{\mathrm{L}}(x)&=\alpha x \qquad x\in[0,1/\alpha]\\ f_{\mathrm{R}}(x)&=[\alpha/(\alpha-1)](1-x) \qquad x\in(1/\alpha,1]\end{aligned} \tag{3}$$

and $S=[0,1/\alpha]$, $S_T=(1/\alpha,1]$.

The function f is nowhere invertible and possesses no stable orbit of a finite period (especially no stable fixed point), so it is a good candidate for chaotic behaviour. Intermittency occurs for values of α slightly greater than one. A typical trajectory $(x_0, x_1, x_2, \ldots)$ shows long phases of monotonic growth according to a power law $(\sim\alpha^n)$. The duration of a laminar phase is longer, the nearer to the origin this phase begins. This behaviour is interrupted by short turbulent oscillations around the unstable fixed point and jumps to smaller values of x (figure 2).

By the prescription

$$s(x)=\begin{cases}0 & 0\leqslant x\leqslant 1/\alpha\\ T & 1/\alpha<x\leqslant 1\end{cases} \tag{4}$$

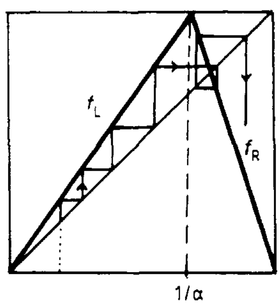

Figure 1. The piecewise linear function $f(x)$.

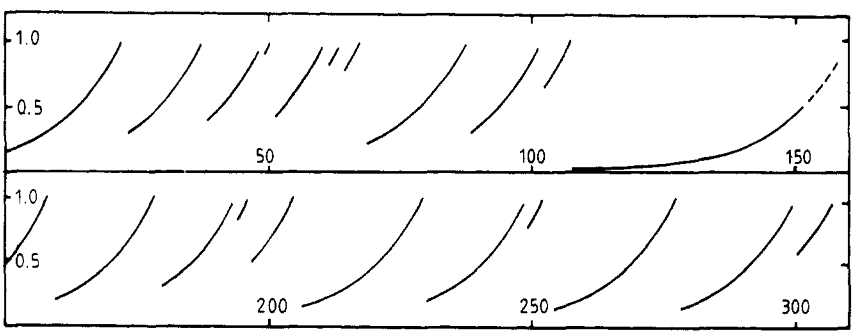

Figure 2. A typical trajectory $x_0, x_1, x_2, \ldots$ ($\alpha=1.1$, $x_0=0.12$). The points of the laminar phases are connected by a smooth curve.

we can construct a corresponding symbolic dynamics, its trajectories being sequences of 0 and T (0 = 'quiet', 'laminar', T = 'turbulent'). A typical trajectory looks like

$$0\ldots 0\,\underbrace{TT\ldots T}_{\lambda_1}\,\underbrace{00\ldots 0}_{l_1}\,\underbrace{TT\ldots T}_{\lambda_2}\,\underbrace{00\ldots 0}_{l_2}\,T\ldots \quad \ldots$$

(cf figure 2) where we have denoted the number of successive 0 by $l_1, l_2, \ldots$, and the number of successive T by $\lambda_1, \lambda_2, \ldots$.

Definition 1. The sequence of successive 0 (T) after the kth change $T0$ ($0T$) is called the kth 0 phase (T phase) and the number l_k (λ_k) of 0 (T) occurring is called the length of the kth 0 phase (T phase).

Definition 2. A value x_n (within a given trajectory) is called the starting point for a 0 phase if $s(x_{n-1}) = T$, $s(x_n) = 0$ (and similarly for a T phase).

In the following we want to consider mainly the behaviour of the lengths of the ('laminar') 0 phases.

Lemma 1. (a) If x_n is the starting point for a 0 phase, then it holds that

$$x_{n-1} \in [1-(\alpha-1)/\alpha^2, 1] \qquad x_n \in [0, 1/\alpha]. \tag{5}$$

(b) If $x_n = x$ is the starting point for a 0 phase, then the corresponding length l_k will be a function $l(x)$ of the starting point,

$$l(x) = \mathrm{int}\left(-\frac{\ln x}{\ln \alpha}\right) = \sum_{k=1}^{\infty} k 1_{(1/\alpha^{k+1}, 1/\alpha^k]}(x) \tag{6}$$

(where int ( ) denotes the integer part and 1_A denotes the indicator function of a set A).

Proof. By definition, we have $x_n \in S$, $x_{n-1} = f_R^{-1}(x_n) \in f_R^{-1}(S) = [1-(\alpha-1)/\alpha^2, 1]$ (see figure 3) and $l(x) = k$ corresponds to

$$f^{k-1}(x) \leqslant 1/\alpha < f^k(x) \leftrightarrow k \leqslant -\frac{\ln x}{\ln \alpha} < k+1 \leftrightarrow \frac{1}{\alpha^{k+1}} < x \leqslant \frac{1}{\alpha^k}.$$

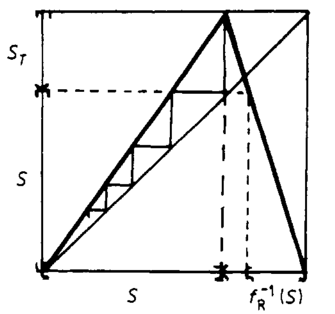

Figure 3. The points in $f_R^{-1}(S)$ are precursors of starting points for 0 phases.

3. Ergodicity of the dynamics

The following properties of our dynamics (1) and (2) are well known and will be cited here for later use.

Theorem 1. (*a*) The transformation f possesses exactly one absolutely continuous invariant (probability) measure, namely the uniform distribution on [0, 1].

(*b*) The dynamics is ergodic with respect to this invariant measure.

Proof. The existence of an invariant measure $P = P \circ f^{-1}$ with density $p(x)$ is ensured by a theorem of Lasota and Yorke (1973), because f is piecewise differentiable with $\inf|f'(x)| > 1$, as the uniqueness of the one hump of f is sufficient (see, for example, Kowalski 1976).

The density $p(x)$ of the invariant measure P must satisfy

$$p(x) = \frac{p(f_{\mathrm{L}}^{-1}(x))}{|f'_{\mathrm{L}}(f_{\mathrm{L}}^{-1}(x))|} + \frac{p(f_{\mathrm{R}}^{-1}(x))}{|f'_{\mathrm{R}}(f_{\mathrm{R}}^{-1}(x))|} = \frac{1}{\alpha} p(f_{\mathrm{L}}^{-1}(x)) + \left(1 - \frac{1}{\alpha}\right) p(f_{\mathrm{R}}^{-1}(x)) \tag{7}$$

which gives $p(x) = 1_{[0,1]}(x)$ directly.

Remark. Other invariant measures are of the type $P = (1/n)\sum_{i=1}^{n} \delta_{x_i}$, where $(x_1, x_2, \ldots, x_n)$ is an (unstable!) orbit of period n, but all these measures are supported by sets of Lebesgue-measure zero.

Corollary 1. Let $F:[0, 1] \to R^1$ be a measurable function. Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} F[f^i(x_0)] = \int_0^1 F(x)p(x)\,\mathrm{d}x \qquad \text{with } p(x) \equiv 1 \tag{8}$$

for almost all x_0.

Remark. This is the famous ergodic theorem stating the equality of the time average $\bar{F}$ over a trajectory (starting with x_0)

$$\bar{F} := \lim_{N\to\infty} \bar{F}^N \qquad \bar{F}^N(x_0) := \frac{1}{N} \sum_{i=0}^{N-1} F[f^i(x_0)]$$

and the ensemble average $E_P F$ of a function $F(\xi)$, where ξ is a random variable distributed according to the probability measure P with density $p(x)$,

$$E_P F(\xi) = \int_0^1 F(x) \cdot p(x)\,\mathrm{d}x$$

for almost all trajectories, i.e. we have

$$\bar{F}^N(x_0) \xrightarrow[N\to\infty]{} E_P F(\xi)$$

for almost all x_0.

Special cases of interest are $F(x) = x$, $F(x) = (x - \bar{x})\,(f^k(x) - \bar{x})$ and $F(x) = 1_A(x)$,
$:[0, 1] \to R^1$ be a measurable function. Then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} F[f^i(x_0)] = \int_0^1 F(x)p(x)\,\mathrm{d}x \qquad \text{with } p(x) \equiv 1 \tag{8}$$

for almost all x_0.

Remark. This is the famous ergodic theorem stating the equality of the time average $\bar{F}$ over a trajectory (starting with x_0)

$\bar{F} := \lim_{N\to\infty} \bar{F}^N$ $\bar{A}$ being a Borel subset of [0, 1].

Assertion. For the asymmetric tent map (1), (2) we have

$$\bar{x} = \tfrac{1}{2} \qquad \overline{(x - \bar{x})(f^k(x) - \bar{x})} = \tfrac{1}{12}[(2 - \alpha)/\alpha]^k \qquad k = 0, 1, 2, \ldots. \tag{9}$$

Proof. The first part is obvious, as P is the uniform distribution on $[0, 1]$, and the second part follows by partial integration and complete induction. The essential step is the calculation ($p(x) \equiv 1$!)

$$\begin{aligned}\overline{xf^k(x)} &= \int_0^1 xf^k(x)\,\mathrm{d}x \\ &= \int_0^1 \frac{y}{\alpha} f^{k-1}(y)\frac{1}{\alpha}\,\mathrm{d}y + \int_0^1 \left[1-\left(1-\frac{1}{\alpha}\right)y\right] f^{k-1}(y)\left(1-\frac{1}{\alpha}\right)\mathrm{d}y \\ &= \frac{2-\alpha}{\alpha}\int_0^1 yf^{k-1}(y)\,\mathrm{d}y + \left(1-\frac{1}{\alpha}\right)\int_0^1 f^{k-1}(y)\,\mathrm{d}y \\ &= \left(\frac{2-\alpha}{\alpha}\right)\overline{xf^{k-1}(x)} + \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\end{aligned}$$

which gives directly

$$\overline{xf^k(x)} - \bar{x}^2 = \left(\frac{2-\alpha}{\alpha}\right)[\overline{xf^{k-1}(x)} - \bar{x}^2] \qquad k = 1, 2, \ldots.$$

This result was numerically found by Grossmann (1983).

4. Ergodicity of the sequence of lengths (of the 0 phases)

The results in this section are formulated for somewhat general functions f satisfying the following properties (which are by no means minimal):

(i) $f(x) = f_L(x)$ strictly increasing in $S = [0, x_0]$,
(ii) $f(x) = f_R(x)$ strictly decreasing in $S_T = (x_0, 1]$,
(iii) $f_L(0) = f_R(1) = 0$, $f_L(x_0) = f_R(x_0) = 1$,
(iv) $f_L \in C^1[0, x_0]$, $f_R \in C^1[x_0, 1]$ with $|f'(x)| > 1$ in $[0, 1]$.

If in analogy to (4) we put

$$s(x) = \begin{cases} 0 & x \in S \\ T & x \in S_T \end{cases} \tag{4'}$$

then $S \subset [0, 1]$ is the set of possible starting points for 0 phases.

Lemma 2. Let $\varkappa(x) = 1_{f_R^{-1}(S)}(x)\, f(x)$. Then we have $\varkappa(x_{n-1}) = x_n$ if x_n is a starting point for a 0 phase, and 0 otherwise. Let $p(x)$ be the density of the measure P being invariant under f. Then, under the assumptions (i)-(iv) above, we have

$$\overline{1_A(\varkappa(x))} = \mathcal{N}\int_{A\cap S} \frac{p(f_R^{-1}(y))}{|f_R'(f_R^{-1}(y))|}\,\mathrm{d}y \qquad \text{for almost all } x_0 \tag{10}$$

with

$$\mathcal{N}^{-1} := \int_S \frac{p(f_R^{-1}(y))}{|f_R'(f_R^{-1}(y))|}\,\mathrm{d}y.$$

Proof. x_n is a starting point if $x_n = f(x_{n-1}) \in S$, $x = x_{n-1} \in f_R^{-1}(S)$. Hence in the invariant case the 'precursors' x of starting points are distributed according to $P|_{f_R^{-1}(S)}$ and consequently the starting points $y = f_R(x)$ themselves are distributed according to $P_s = P|_{f_R^{-1}(S)} \circ f_R^{-1}$, i.e. they have the density

$$p_s(y) = \mathcal{N} \frac{p(f_R^{-1}(y))}{|f'_R(f_R^{-1}(y))|} 1_S(y) \tag{11}$$

with the normalisation factor given by (10). Integration over A and the ergodicity of the process (8) give the desired result. Let $n(N) = n(N(x_0))$ be the total number of 0 phases within a trajectory $(x_0, x_1, \ldots, x_N)$, where for convenience N is chosen to finish a 0 phase. Then $n(N)$ is the number of changes $T0$, and so we have

$$n(N) = \sum_{i=0}^{N-1} 1_{f_R^{-1}(S)}(x_i) \qquad x_i = f^i(x_0). \tag{12}$$

Lemma 3. We have

$$\lim_{N\to\infty} \frac{n(N)}{N} = \int_0^1 1_S(y) \frac{p(f_R^{-1}(y))}{|f'_R(f_R^{-1}(y))|} \, dy \qquad \text{for almost all } x_0. \tag{13}$$

Proof. f is ergodic with respect to $p(x)$, so we have

$$\lim_{N\to\infty} \frac{n(N)}{N} = \int_0^1 1_{f_R^{-1}(S)}(x) p(x) \, dx$$

which together with $y = f_R(x)$ proves the lemma.

Now the following problem arises. Suppose ϕ to be a real function defined on the natural numbers and let $l_1, l_2, \ldots, l_{n(N)}, \ldots$, be the sequence of lengths of 0 phases. Let us now consider the empirical mean value

$$\overline{\phi(l)}^n := \frac{1}{n(N)} \sum_{k=1}^{n(N)} \phi(l_k). \tag{14}$$

Theorem 2. Let ϕ: $\mathbb{N} \to R^1$ be an arbitrary function. Then, under the assumptions (i)-(iv) on f, we have for almost all x_0

$$\lim_{n\to\infty} \overline{\phi(l)}^n = \int_0^1 \phi[l(y)] p_s(y) \, dy. \tag{15}$$

Proof. By definition

$$\overline{\phi(l)}^n = \frac{1}{n(N)} \sum_{i=0}^{N-1} \phi[l(f(x_i))] 1_{f_R^{-1}(S)}(x_i)$$

so we have

$$\lim_{n\to\infty} \overline{\phi(l)}^n = \lim_{N\to\infty} \overline{\phi(l)}^{n(N)} = \lim_{N\to\infty} \frac{N}{n(N)} \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \phi[l(f(x_i))] 1_{f_R^{-1}(S)}(x_i).$$

Because of the ergodicity of f the second limit is equal to the ensemble average

$$\int_0^1 1_{f_R^{-1}(S)}(x) \phi[l(f(x))] p(x) \, dx.$$

By substituting $y = f_R(x)$, lemma 3 and the invariant density of starting points $p_s(x)$ from lemma 2, we obtain the theorem.

Corollary 2. We have (for almost all x_0)

$$\overline{1_{\{m\}}(l)} = \int_0^1 1_{\{m\}}[l(y)]p_s(y)\,\mathrm{d}y \equiv P_S\{1(\eta)=m\} \qquad m=1,2,\ldots \tag{16}$$

$$\overline{l^k} = \int_0^1 l^k(y)p_s(y)\,\mathrm{d}y \equiv E_{P_s}l^k(\eta) \tag{17}$$

and in particular

$$\bar{l} = \int_0^1 l(y)p_s(y)\,\mathrm{d}y \equiv E_{P_s}l(\eta).$$

Proof. The result follows immediately from theorem 2 if ϕ is taken to be $\phi(l)=1_{\{m\}}(l)$ or $\phi(l)=l^k$ (especially $\phi(l)=l$).

Remark. The sequence $(l_1, l_2, \ldots, l_{n(N)}, \ldots)$ of lengths is therefore ergodic in the following sense.

(i) The empirical mean length $\bar{l}^n$ calculated from a long trajectory as $n\to\infty$ tends to the expectation value El of a random variable $l(\eta)$, η being absolutely continuously distributed with density $p_s(y)$.

(ii) The relative frequency of a 0 phase of fixed length m calculated from a long trajectory as $n\to\infty$ tends to the probability of the discrete random variable $l(\eta)$ taking the value m.

The next problem is to investigate the correlations of consecutive (or more distant) lengths of 0 phases. Let $\tau(l_k)=l_{k+1}$ be the shift of the counting of the 0 phases for one step. Then the value

$$\overline{l\tau(l)}^n := \frac{1}{n(N)}\sum_{k=1}^{n(N)} l_k l_{k+1} \tag{18}$$

(or better its centred expression) is characteristic for the interdependence of consecutive lengths.

Lemma 4. We have

$$\overline{l\tau(l)}^n = \frac{1}{n(N)}\sum_{i=0}^{N-1} 1_{f_R^{-1}(S)} l[f(x_i)]\, l[f^{\,l[f(x_i)]+f^{\,l[f(x_i)]+1}(x_i)+1}](x_i). \tag{19}$$

Proof. Let x_i be the last symbol in a T phase and $f(x_i)$ the value corresponding to the first 0 in the kth 0 phase with length $l_k=l(f(x_i))$. Then $f^{l_k}(x_i)$ creates the last symbol 0 and $f^{l_k+1}(x_i)$ is the starting point for a T phase of length λ_{k+1} (λ_{k+1} being a function $\lambda(y)$ of the new starting point $y=f^{l_k+1}(x_i)+1$ and therefore a function of x_i). Hence $f^{l_k+\lambda_k+1}(x_i)$ is the starting point for the $(k+1)$th 0 phase.

Theorem 3. We have for almost all x_0

$$\overline{l\tau(l)} \equiv \lim_{n\to\infty}\frac{1}{n(N)}\sum_{k=1}^{n(N)} l_k l_{k+1} = \int_0^1 l(x)(\tau\circ l)(x)p_s(x)\,\mathrm{d}x. \tag{20}$$

Proof. The result is a consequence of theorem 2, if one uses lemma 4 and the fact that $\tau \circ l$ is a function of the (first) starting point x. (Here $\tau \circ l$ denotes the length l_{k+1} of the $(k+1)$th 0 phase as a function of the starting point x of the kth 0 phase.)

Remark. Theorem 3 can be further generalised in the sense that for an arbitrary function $\psi : \mathbb{N}^{r+1} \to R^1$ it holds that

$$\overline{\psi(l, \tau \circ l, \ldots, \tau^r \circ l)} = \lim_{n\to\infty} \frac{1}{n(N)} \sum_{k=0}^{n(N)} \psi(l_k, l_{k+1}, \ldots, l_{k+r})$$

$$= \int_0^1 \psi[l(x), (\tau \circ l)(x), \ldots, (\tau^r \circ l)(x)] p_s(x)\, \mathrm{d}x$$

$$\equiv E_{P_s} \psi[l(\eta), (\tau \circ l)(\eta), \ldots, (\tau^r \circ l)(\eta)] \tag{21}$$

for almost all x_0.

5. Results for the piecewise linear map

Throughout this section f is assumed to be the piecewise linear function (2).

Lemma 5. The invariant distribution P_s of the starting points is absolutely continuous with the density

$$p_s(x) = \begin{cases} \alpha & 0 \leqslant x \leqslant 1/\alpha \\ 0 & 1/\alpha < x \leqslant 1. \end{cases} \tag{22}$$

Proof. We have $p(x) \equiv 1$ and $|f'_{\mathrm{R}}(f_{\mathrm{R}}^{-1}(y))| = \alpha/(\alpha - 1)$ independent of y, so by lemma 2 and (11) P_s turns out to be the uniform distribution restricted to $[0, 1/\alpha]$.

Theorem 4. We have

$$E_{P_s} \psi[l(\eta), (\tau \circ l)(\eta), \ldots, (\tau^r \circ l)(\eta)]$$

$$= \sum_{k_0=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \left(\psi(k_0, \ldots, k_r) \prod_{i=0}^{r} \frac{\alpha - 1}{\alpha^{k_i}} \right) \tag{23}$$

for the piecewise linear function (2). To prove this theorem we use the function $\lambda(x)$ giving the length of a T phase starting with the point x.

Lemma 6. It holds that

$$\lambda(x) = \sum_{m=1}^{\infty} m 1_{I_m}(x) \tag{24}$$

with

$$I_m := \begin{cases} [y_1, 1] & m = 1 \\ (y_{m-2}, y_m] & m = 2, 4, 6, \ldots;\ y_m = f_{\mathrm{R}}^{-m}(y_0),\ y_0 = 1/\alpha \\ [y_{m+2}, y_m] & m = 3, 5, 7, \ldots. \end{cases} \tag{25}$$

Proof. Let x be a starting point for a T phase. Then we have $\lambda(x)=m$ if and only if

$$x\in I_m=\{x|x\in S_T, f_{\mathrm{R}}(x)\in S_T,\ldots,f_{\mathrm{R}}^{m-1}(x)\in S_T, f_{\mathrm{R}}^m(x)\in S\}$$

(figure 4), because we have

$$f_{\mathrm{R}}^k(I_m)=I_{m-k}\subset S_T \qquad k=0,1,\ldots,m-1$$

$$f_{\mathrm{R}}^m(I_m)=\begin{cases} S & m=1\\ S\backslash\{0\} & m=2,3,\ldots\end{cases}$$

and $\lambda(x_n)=m$ if $x_n\in I_m$, $x_{n-1}\in S$ only for these intervals.

Proof of theorem 4. To simplify the notation we define

$$L(x):=F(\alpha^{l(x)}x) \text{ with } F(x):=\sum_{m=1}^{\infty} f_{\mathrm{R}}^m(x)1_{I_m}(x). \tag{26}$$

Then it follows that

$$(\tau\circ l)(x)=l[L(x)]=\mathrm{int}\left(-\frac{\ln F(\alpha^{\mathrm{int}(-\ln x/\ln\alpha)}x)}{\ln\alpha}\right)$$

and, more generally, $(\tau^m\circ l)\ (x)=(l\circ L^m)\ (x)$, $m=1,2,\ldots$. Using lemma 5 and the given definitions we can write

$$\begin{aligned}
E\psi\equiv E_{P_s}\psi[l(\eta),\ldots,(\tau^r\circ l)(\eta)]&=\alpha\int_0^{1/\alpha}\psi[l(x),(l\circ L)(x),\ldots,(l\circ L^r)(x)]\,\mathrm{d}x\\
&=\alpha\int_0^{1/\alpha}\psi\left[\mathrm{int}\left(-\frac{\ln x}{\ln\alpha}\right),(l\circ F)(\alpha^{\mathrm{int}(-\ln x/\ln\alpha)}x),\ldots,\right.\\
&\qquad\left.\times(l\circ L^{r-1}\circ F)(\alpha^{\mathrm{int}(-\ln x/\ln\alpha)}x)\right]\mathrm{d}x\\
&=\alpha\sum_{k_0=1}^{\infty}\int_{\alpha^{1/k_0+1}}^{\alpha^{1/k_0}}\psi[k_0,(l\circ F)(\alpha^{k_0}x),\ldots,(l\circ L^{r-1}\circ F)(\alpha^{k_0}x)]\,\mathrm{d}x.
\end{aligned}$$

The subsequent substitutions $y=\alpha^k x$, $z=f_{\mathrm{R}}^m(y)$ yield

$$\begin{aligned}
E\psi&=\sum_{k_0=1}^{\infty}\frac{1}{\alpha^{k_0}}\int_{1/\alpha}^{1}\sum_{m=1}^{\infty}\psi[k_0,(l\circ f_{\mathrm{R}}^m)(y),\ldots,(l\circ L^{r-1}\circ f_{\mathrm{R}}^m)(y)1_{I_m}(y)\,\mathrm{d}y\\
&=\alpha\sum_{k_0=1}^{\infty}\frac{1}{\alpha^{k_0}}\sum_{m=1}^{\infty}\left(\frac{\alpha-1}{\alpha}\right)^m\int_0^{1/\alpha}\psi[k_0,l(z),\ldots,(l\circ L^{r-1})(z)]\,\mathrm{d}z
\end{aligned}$$

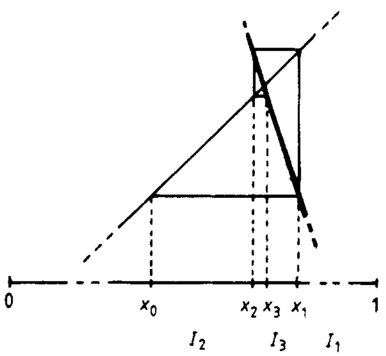

Figure 4. Points starting in $I_1, I_2, I_3, \ldots$, generate T phases of length $1, 2, 3, \ldots$.

$$= \alpha \sum_{k_0=1}^{\infty} \frac{\alpha-1}{\alpha^{k_0}} \int_0^{1/\alpha} \psi[k_0, l(z), \ldots, (l \circ L^{r-1})(z)]\, dz.$$

Iterating these substitutions, we finally obtain

$$E\psi = \sum_{k_0=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \left(\prod_{i=0}^{r} \frac{\alpha-1}{\alpha^{k_i}} \psi(k_0, \ldots, k_r) \right).$$

Corollary 3. (i) The relative frequency of the lengths with value m as $n \to \infty$ (for almost all x_0) tends to a geometric distribution with

$$P_s\{l(\eta) = m\} = (\alpha - 1)/\alpha^m \qquad m = 1, 2, \ldots. \tag{27}$$

(ii) The empirical mean length $\bar{l}^n$ and the empirical variance $(\bar{\sigma}^n)^2 = \overline{(l - \bar{l}^n)^2}^n$ as $n \to \infty$ (for almost all x_0) tend to the values

$$El(\eta) = \alpha/(\alpha - 1) \qquad Dl(\eta) \equiv E(l - El)^2 = \alpha/(\alpha - 1)^2 \tag{28}$$

and in particular for the relative empirical dispersion we have

$$\lim_{n\to\infty} \frac{\bar{\sigma}^n}{\bar{l}^n} = \frac{\sqrt{Dl}}{El} = \frac{1}{\sqrt{\alpha}} \qquad \text{for almost all } x_0.$$

(iii) For almost all x_0 we have

$$\overline{1_{\{m_0,\ldots,m_r\}}(l, \tau \circ l, \ldots, \tau^r \circ l)} = \prod_{i=0}^{r} \overline{1_{\{m_i\}}(\tau^i \circ l)}$$

or, by ergodicity,

$$P_s\{l(\eta) = m_0, \ldots, (\tau^r \circ l)(\eta) = m_r\} = \prod_{i=0}^{r} P_s\{(\tau^i \circ l)(\eta) = m_i\}. \tag{29}$$

Proof. (i) The statement follows immediately from theorem 4, choosing $\psi = 1_{\{m\}}$.

(ii) The results are well known facts on geometrical distributions (taking into account that $m = 0$ is forbidden by definition).

(iii) For

$$\psi(k_0, \ldots, k_r) = 1_{\{m_0,\ldots,m_r\}}(k_0, \ldots, k_r) = \prod_{i=0}^{r} 1_{\{m_i\}}(k_i)$$

(i) yields directly

$$P_s\{l(\eta) = m_0, \ldots, (\tau^r \circ l)(\eta) = m_r\} = \prod_{i=0}^{r} \left(\frac{\alpha-1}{\alpha^{m_i}} \right) = \prod_{i=0}^{r} P\{(\tau^i \circ l)(\eta) = m_i\}.$$

Remark. Part (iii) of the corollary implies that the conditional relative frequency of lengths following a length of value m in a long trajectory is the same as the unconditional frequency and all correlations vanish identically,

$$\overline{(l - \bar{l})(\tau^m \circ l - \bar{l})} = \mathrm{Cov}_{P_s}(l(\eta), (\tau^m \circ l)(\eta)) = 0 \qquad m = 1, 2, \ldots. \tag{30}$$

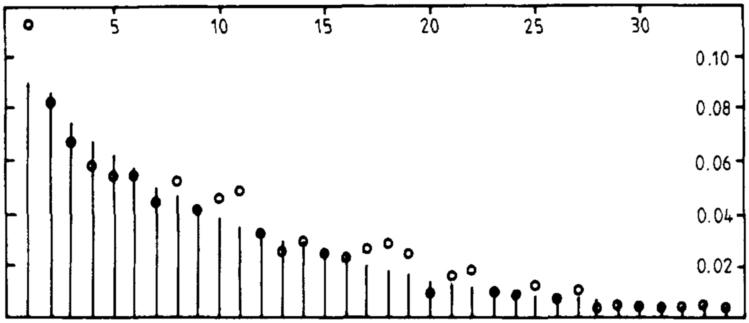

Figure 5. Relative frequency of the lengths of 0 phases (circles) compared with the exact geometrical distribution (straight lines) for $\alpha = 1.1$ and 10 000 lengths.

Table 1. Comparison of empirical moments and asymptotically exact values calculated by the invariant distribution for different values of α and 10 000 0 phases. (The subsequent rows list mean value, dispersion, relative dispersion and correlation of subsequent states.)

n	α	$\bar{l}^n$	El	$\bar{\sigma}^n$	$\sqrt{Dl}$	$\frac{\bar{\sigma}^n}{\bar{l}^n}$	$\frac{\sqrt{Dl}}{El}$	$\overline{(l-\bar{l})(\tau\circ l-\bar{l})}$	$C(l, \tau\circ l)$
10 000	1.2	6.0	6	5.5	5.5	0.92	0.91	0.0	0
10 000	1.1	10.9	11	10.2	10.5	0.94	0.95	0.3	0
10 000	1.08	13.3	13.5	13.1	13.0	0.99	0.96	3.0	0
10 000	1.06	17.8	17.7	17.6	17.2	0.99	0.97	−2.5	0
10 000	1.04	26.2	26	25.5	25.5	0.97	0.98	−3.8	0

6. Numerical results

The dynamics (1) and (2) can be easily simulated on a small pocket calculator. In figure 5 the empirical relative frequencies of the lengths of 0 phases are shown after realising 10 000 lengths (i.e. $n(N) = 10\,000$, $N \approx 10\,000\,\bar{l}$) for a value $\alpha = 1, 1$. That gives a good agreement with the corresponding probabilities (27) of a geometrical distribution.

Table 1 shows the comparison of empirical means ($n(N) = 10\,000$) (iteration with 14 digits) and asymptotically exact values which again agree within the expected accuracy.

For visualising possible correlations we have plotted l_{k+1} against l_k. The result is shown in figure 6. The cloud of points underlines that there is no obvious functional dependence $l_{k+1} = g(l_k)$ on the number l_k (but only a dependence in the form $l_{k+1}(x) = (\tau \circ l_k)(x)$), in particular, a fixed value $l_k = m$ may be followed by all possible values of l_{k+1}. This is a consequence of the fact that consecutive lengths are totally uncorrelated.

7. Summary and interpretation

The main result for the asymmetric tent map is formulated in corollary 3. Obviously, our model system described by the dynamics (1) and (2) shows an intermittent behaviour for $\alpha = 1 + \varepsilon$, $\varepsilon \ll 1$, where at apparently random times long phases of regular growth

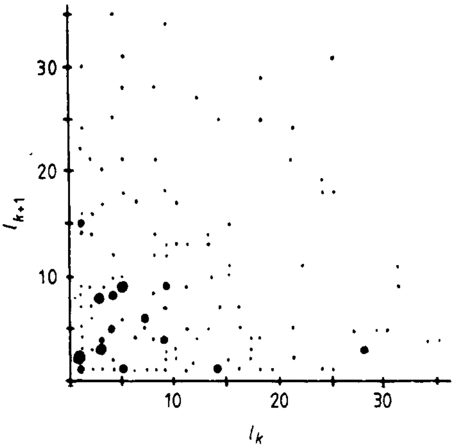

Figure 6. Empirical correlations of consecutive lengths (plot of l_{k+1} against l_k for $k = 1, \ldots, 150$, $\alpha = 1.1$).

$(\sim\alpha^n)$ are interrupted by (monotonic or oscillating) irregular jumps. Correspondingly, a symbolic dynamics shows long phases of 0 apparently randomly interrupted by short T phases.

If one realises numerically a sufficiently long trajectory and analyses the lengths of the 0 phases then almost certainly (with respect to the initial point) the following behaviour occurs.

(i) The relative frequencies of the possible lengths asymptotically become geometrically distributed. If a trajectory is stopped at an arbitrary time $n \gg 1$, then the next length appears as a geometrically distributed random variable.

(ii) Consecutive lengths are independent in the sense that the knowledge of the value l_k does not allow any prediction of l_{k+1} (possessing more information than that given in (i)).

(iii) The mean length becomes larger and larger as $\alpha \downarrow 1$, but the relative dispersion remains finite, so the randomness is also preserved with reference to the absolute length.

The lengths of the turbulent phases can, of course, be analysed in the same way, and they turn out to be geometrically distributed as well:

$$P_\sigma\{\lambda(\vartheta) = m\} = \frac{1}{\alpha - 1}\left(\frac{\alpha - 1}{\alpha}\right)^m \qquad m = 1, 2, \ldots$$

$$E\lambda(\vartheta) = \alpha \qquad D\lambda(\vartheta) = \alpha(\alpha - 1)$$

where ϑ is a random variable distributed according to P_σ with density $p_\sigma(x) = [\alpha/(\alpha - 1)]1_{S_T}(x)$ being the density of starting points for a T phase.

To prove this we must use from lemma 6 that $\lambda(x) = m$ if and only if $x \in I_m$, where subsequent intervals $I_0, I_1, \ldots,$ are scaled down by a factor $1/\alpha$.

The assertions on the ergodic properties of the sequence of lengths $\{l_k, \lambda_k\}_{k=1}^{\infty}$ are applicable to the more general functions f satisfying the assumptions at the start of § 4, but the analytical calculation may be more complicated.

A comparison of our results with those obtained in the literature for non-linear maps seems to be not directly possible, but we believe that intermittency of type III (Schuster 1984) occurs. Writing $\alpha = 1 + \varepsilon$, $\varepsilon \ll 1$, we observe from (27) and (28) the power law for the mean length $\bar{l} \sim \varepsilon^{-1}$ and the exponential decay for the probabilities

$$p_m := P_s\{l(\eta) = m\} \sim \mathrm{e}^{-\varepsilon m} \qquad \text{as } \varepsilon \downarrow 1$$

which should still hold for general type-III systems. (In Schuster (1984) type III is introduced by $x_{n+1} = -(1+\varepsilon)x_n - ux_n^3$, hence for $\varepsilon \ll 1$ every second iterate satisfies $x_{n+2} = (1+2\varepsilon)x_n + \ldots$ and causes an additional factor of two for the probabilities p_m.) In our opinion the model clearly shows how macroscopic randomness can arise from microscopically deterministic dynamics.

Acknowledgment

The authors would like to thank one of the referees for his detailed comments which led to a refined version of the manuscript.

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